

**Hale School**

**MATHEMATICS**

**SPECIALIST**

**3CD**

**Semester Two Examination 2011**

**MARKING KEY and SOLUTIONS**

**Section Two**

**Calculator-Assumed**

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**Question 8 [7 marks]**

The current I(t) amperes, in an electrical circuit, t milliseconds after a switch is turned on, obeys the differential equation :

 amperes/millisecond

Initially there is no current flowing when the switch is turned on.

(a) Give the initial rate of change of the current when the switch is turned on.

[1]

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| **Solution** |
| At t = 0, I = 0 Hence |
| **Specific Behaviours** |
| ✓ Correct value using the derivative |

(b) Using the separation of variables technique, determine the defining rule for I(t) in terms of t.

[4]

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| **Solution** |
| Hence -0.5 ln | 5 – 2I | = t + c  ln | 5 – 2I | = -2t + k  As 5 – 2I > 0, 5 – 2I = e-2t. ek  Using I(0) = 0, 5 - 2(0) = e0. ek,  Hence ek = 5  5 - 2I = 5e-2t  Gives I(t) = 2.5 – 2.5 e-2t |
| **Specific Behaviours** |
| ✓ Separates variables correctly  ✓ Anti-differentiates correctly with an integration constant  ✓ Correct evaluation of integration constant  ✓ Obtains I(t) explicitly as a function of t |

(c) Explain what is happening to the current after 1 second.

[2]

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| **Solution** |
| At t = 1000, I = 2.5 - 2.5 e-2000 = 2.5 amperes  Hence after 1 second, the current has reached its maximum value and is not changing. |
| **Specific Behaviours** |
| ✓ Correct t value, t = 1000  ✓ Comments that the current is not changing at all (reached some equilibrium) |

**Question 9 [8 marks]**

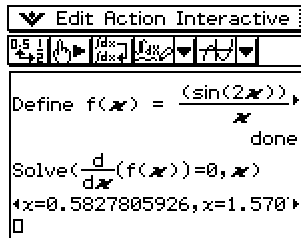
A

The curve given by sin2(2x) = xy is shown.

(a) Find an expression for  in terms of x and y using implicit differentiation.

[4]

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| **Solution** |
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| **Specific Behaviours** |
| ✓✓ Derivative of sin2(2x)  ✓ Derivative of xy  ✓ Obtains dy/dx as the subject in terms of x and y |

(b) Give the x co-ordinate for point A, the global maximum of the curve, correct to 0.01.

[2]

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| **Solution** |
| For global maximum require dy/dx = 0  From CAS x = 0.58 (2 d.p.) |
| **Specific Behaviours** |
| ✓ Condition for the global maximum  ✓ Value for x correct to 0.01 |

(c) Determine the slope of the curve as x 🡪 0.

[2]

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| **Solution** |
| As x 🡪 0,  = 2(4) - 4  = 4  Hence the slope of the curve approaches 4 as x 🡪 0 |
| **Specific Behaviours** |
| ✓ Use of the special trigonometric limit involving the sine function  ✓ Correct value |

**Question 10 [10 marks]**

Using Calculus techniques, find the following indefinite integrals :

(a) 

[2]

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| **Solution** |
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| **Specific Behaviours** |
| ✓ Higher power and division by 3/2  ✓ Divide by 2 to undo the chain rule |

(b) 

[2]

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| **Solution** |
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| **Specific Behaviours** |
| ✓ Natural logarithm anti-derivative using absolute value  ✓ Divide by 4 to undo the chain rule |

(c) 

[2]

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| **Solution** |
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| **Specific Behaviours** |
| ✓ Recognise 4x is related to the derivative factor, so we can anti-differentiate  ✓ Correct answer |

(d)  Put u = x + 2

[4]

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| **Solution** |
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| **Specific Behaviours** |
| ✓ Substitute for x and dx  ✓ Multiply out integrand in terms of u  ✓ Correct anti-derivative in terms of u  ✓ Express in terms of x |

**Question 11 [6 marks]**

Using the method of proof by induction, prove that for all counting numbers n, the expression 34n - 1 is divisible by 80.

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| **Solution** |
| For n = 1 34 - 1 = 80 True for n = 1  Assume true for n = k i.e. 34k - 1 = 80m for some integer m  Consider for n = k + 1 34(k + 1) - 1 = 34k + 4 - 1  = 34k . 34 - 1  = 81(80m + 1) - 1  = 81(80m) + 81 - 1  = 80(81m + 1)  This means that 34(k + 1) - 1 is divisible by 80.  Hence true for n = k + 1  Hence true for ALL values of n. |
| **Specific Behaviours** |
| ✓ Show true for n = 1  ✓ Assume true for n = k, using some multiple of 80  ✓ Consider expression for n = k + 1 (but not assuming it is divisible by 80)  ✓ Express 34k + 4 in terms of the previous case  ✓ Algebraically show that this expression has a factor of 80  ✓ Explain that since true for n = k + 1 then true for ALL cases |

**Question 12 [8 marks]**

A particle experiencing simple harmonic motion has a displacement of 10 cm when its velocity is cm s-1 and when the particle is at its equilibrium position it has a velocity of 40 cm s-1.

(a) Determine the period and amplitude of its motion.

[5]

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| **Solution** |
| Since SHM assume x(t) = A cos(nt)  Hence v(t) = - An sin(nt)  When x = 10, v =  hence 10 = A cos (nt)  = - An sin(nt)  Using cos2(nt) + sin2(nt) = 1 then  …… (1)  When x = 0, v = 40 hence 40 = - An ….. (2)  Solving gives A = 20  n = 2 Hence Period T = π seconds |
| **Specific Behaviours** |
| ✓ Knowledge of the form for x(t), v(t)  ✓ Writes an equation for x = 10  ✓ Writes equation for x = 0  ✓ Solves for A, n  ✓ Deduces the period T |

(b) Determine the distance travelled by the particle, correct to the nearest 0.1 cm, over any period of time equal to  seconds.

[3]

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| **Solution** |
| Period of motion T = π seconds  Hence time period represents 4.5 cycles of motion.  Distance travelled = 4.5 (4A) (1 cycle involves 4A distance)  = 18A  = 18(20) cm  = 360 cm |
| **Specific Behaviours** |
| ✓ Recognises distance in one cycle = 4A  ✓ Writes expression for required distance  ✓ Calculates distance correctly |

**Question 13 [9 marks]**

(a) Solve the equation z5 = 1 over the set of complex numbers, giving solutions in exact exponential form.

[3]

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| **Solution** |
| z5 = cis 0  Hence z =  where k = 0, 1, 2, 3 or 4  Solutions are z = cis(2π/5), cis(4π/5), cis(6π/5), cis(8π/5), cis(2π)  i.e. z = cis(2π/5), cis(4π/5), cis(-4π/5), cis(-2π/5), 1  Hence z = |
| **Specific Behaviours** |
| ✓ Writes 5 solutions with the correct equally spaced arguments  ✓ Uses exponential form for solutions  ✓ Uses arguments with correct convention -π < Arg (z) ≤ π |

(b) Prove that the sum of the complex roots to part (a) is zero.

[3]

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| **Solution** |
| Let z1 =  = w  The Sum of the roots = w + w2 + w3 + w4 + w5  =  Since w is a solution then w5 = 1  So Sum of roots =  = 0 |
| **Specific Behaviours** |
| ✓ Writes roots in terms of the principal root  ✓ Applies sum of a geometric series  ✓ Uses w5 = 1 idea |

c) Hence, or otherwise, solve exactly the equation z7 + z5 - z2 - 1 = 0.

[3]

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| **Solution** |
| Solve z5(z2 + 1) - (z2 + 1) = 0  (z2 + 1)(z5 – 1) = 0  So z2 + 1 = 0 or z5 - 1 = 0  Hence z = i, - i or |
| **Specific Behaviours** |
| ✓ Factorises correctly and applies Null Factor Theorem  ✓ Solves z2 + 1 = 0 correctly (in any complex form)  ✓ Uses solutions from part (b) |

**Question 14 [10 marks]**

Consider the functions f(x) = xn and g(x) = 

and the area trapped between these curves.

The graph shown below shows the case

where n = 3 i.e. f(x) = x3

(a) Using Calculus, determine the exact area between the curves for the case when n = 3.

[4]

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| **Solution** |
| Intersect when x = 0, x = 1 for all values of n > 0  Area =  =  = |
| **Specific Behaviours** |
| ✓ Intersects at x = 0, x = 1  ✓ Expression for exact area  ✓ Anti-derivative correct  ✓ Correct value |

(b) If the exact area between the curves is  square units, determine the value of n.

[4]

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| **Solution** |
| Intersect when x = 0, x = 1 for all values of n > 0  Area =  =  =  Hence  Solving gives n = 7.5 |
| **Specific Behaviours** |
| ✓ Writes definite integral expression for area  ✓ Correct anti-derivative in terms of n  ✓ Writes equation to solve for n  ✓ Correct value of n |

(c) If we consider an extremely high power for n, explain what happens to the area between the two curves.

[2]

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| **Solution** |
| Consider n 🡪 ∞, A(n) 🡪 2/3  The area becomes a constant value of two-thirds, which is the area under the graph of  the square root function. |
| **Specific Behaviours** |
| ✓ Consider n increasing without bound  ✓ States the area becomes a constant value of 2/3 |

**Question 15 [10 marks]**

A co-ordinate system is defined showing the positive co-ordinate axes with O being the origin. Two part time rock-climbers Des Duller and Kev Krudder are each attached to two straight wires that allow them to slide down within a wide canyon.

At exactly 0930 hours, Des is at a position of -250**i** + 350**j** + 700**k** metres and is sliding down his wire with velocity 2.5**i** – **j** – **k** metres per second. Meanwhile Kev is stationary at a position 500**i** – 200**j** + 800**k** metres admiring the view. At exactly 0935 hours, Kev begins to slide down his wire at a velocity of -0.5**i** + **j** – 1.5**k** metres per second.

D

K

**z**

**x**

**y**

O

(a) What is Kev’s speed (correct to the nearest 0.01 m/sec) as he slides down his wire ?

[1]

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| **Solution** |
| **v**k = -0.5**i** + **j** – 1.5**k**  Kev’s speed is 1.87 m/sec (2 d.p.) |
| **Specific Behaviours** |
| ✓ Gives the correct speed to 0.01 m/sec |

(b) At what angle to the horizontal plane does Kev slide down, correct to the nearest degree ?

[2]

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| **Solution** |
| Horizontal plane normal vector **n** = 0**i** + 0**j** + 1**k**  Angle between **n** and **v**k **:** θ = 143o  Hence angle Kev’s slide and the plane is  90 – θ i.e. -53o  Kev’s slides down at an angle of 53o to the horizontal |
| **Specific Behaviours** |
| ✓ Uses the plane normal vector to determine an angle  ✓ Gives the correct angle |

**Question 15 [contd.]**

(c) It is known that Des and Kev do not collide. Determine the distance of their closest approach (correct to the nearest metre) and when this occurs (correct to the nearest second).

[3]

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| **Solution** |
| At 0935 Des is at **r**D =  Closest approach when D**r**K **.**  D**v**K = 0    13.25t - 700 = 0  t = 52.83… sec after 0935 hrs  Hence they are closest at 0935 and 53 seconds.  Closest approach is 431 metres. |
| **Specific Behaviours** |
| ✓ Finds position for Des at 0935 hrs  ✓ Gives time for closest approach to nearest second  ✓ Gives closest approach to nearest metre |

(d) If Kev was able to select both the speed and the time at which he commenced sliding down his wire, determine the distance, correct to the nearest metre, he would be able to get closest to Des ?

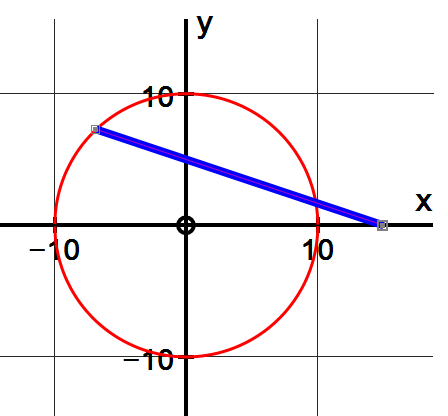
Explain showing your method.

[4]

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| **Solution** |
| The question is equivalent to finding the closest approach of  one line in space to the other line. Equation for each line  (using independent parameters) :  **r**D =  **r**K =  From CAS Closest approach is 49 metres if Kev can adjust  his starting position and speed. |
| **Specific Behaviours** |
| ✓ Explains closest approach between 2 lines in space  ✓ Uses velocities as direction vectors for each equation  ✓ Uses independent parameters for the lines  ✓ Closest approach correct to nearest metre |

**Question 16 [12 marks]**

A small girl G is riding on a merry-go-round and her mother is watching from a fixed position at point M that is 5 metres from the merry go-round. The merry-go-round rotates at a speed of 1 revolution every 10 seconds in an anti-clockwise direction.



**G**

**M**

Let G have co-ordinates (x, y) at any time t seconds after the start. Assume that the girl begins the merry-go-round ride at (10, 0), the closest position to the mother.

(a) Given that we can use parametric equations for the position of the point G in the form of :

x = a cos bt (metres)

y = a sin bt

Explain why a = 10 and b =  .

[2]

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| **Solution** |
| Largest value of x and y is 10 since the radius of the merry-go-round is 10 metres, so  a = 10. 2π radians per 10 seconds = π/5 = b (angular speed in radians) |
| **Specific Behaviours** |
| ✓ Explains that a = radius of circle  ✓ Explains that b = angular speed in radians |

(b) If the mother is directly east of the merry-go-round, at what rate, correct to the nearest cm/second, is the girl travelling in a westerly direction 3 seconds after the start of the ride ?

[3]

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| **Solution** |
| Speed in a westerly direction is given by dx/dt.  dx/dt = -10 sin (3π/5).(π/5) = -5.9756 … cm/sec  Hence the girl is travelling WEST at a rate of 5.98 m/sec after 3 seconds. |
| **Specific Behaviours** |
| ✓ Uses dx/dt to answer question  ✓ Differentiates correctly to find dx/dt  ✓ Gives answer correct to 2 d.p. |

**Question 16 (contd.)**

The distance that the girl is from the mother will fluctuate during the merry-go-round ride.

We can define S(t) = the distance MG (distance between the girl from the mother)

(c) Express S(t) as a function of t.

[1]

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| **Solution** |
| Girl is at (10cos πt/5, 10 sin πt/5), Mother is at (15,0)  S(t) = |
| **Specific Behaviours** |
| ✓ Uses distance formula correctly to express S(t) |

(d) Determine the rate, correct to the nearest cm/second, at which the distance between the small girl and the mother is changing after 3 seconds.

[2]

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| **Solution** |
| Using CAS : dS/dt = 5.79 m/sec  Hence the girl is moving away from the mother  at a rate of 4.39 m/sec. |
| **Specific Behaviours** |
| ✓ Recognises that the derivative S’(t) needs to be found.  ✓ Gives answer correct to 2 d.p. |

(e) Determine when the small girl is moving away from the mother at the fastest rate, correct to the nearest 0.01 seconds.

[2]

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| **Solution** |
| Small girl is moving away at the fastest rate when  S’(t) is a maximum.  Using CAS t = 1.34 seconds |
| **Specific Behaviours** |
| ✓ Recognises that the maximum for S’(t) needs to be found  ✓ Gives answer correct to 2 d.p. |

**Question 16 (contd.)**

(f) How far is the small girl from the mother at the moment she is moving away from the mother at the fastest rate ?

[1]

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| **Solution** |
| Using CAS S(1.338….) = 11.180 ….  Hence when the small girl is moving away at the fastest rate, the girl is 11.18 m away. |
| **Specific Behaviours** |
| ✓ Gives answer correct to 2 d.p. |

(g) Explain the geometric significance of the position of the small girl when she is moving away from the mother at the fastest rate.

[1]

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| **Solution** |
| The small girl is moving away at the fastest rate from the mother, at the instant when the line segment connecting the mother to the girl is a TANGENT to the circle.  Note : MG2 + 102 = 152  [Right triangle dimensions] |
| **Specific Behaviours** |
| ✓ Explains that the MG is a tangent to the circle. |